JOURNAL OF COMPUTATIONAL PHYSICS 60, 155–160 (1985)

Note

Natural Convection at Very High Rayleigh Numbers

The ENTWIFE finite-element code has been applied successfully to the simulation of a wide range of convective flows in recent years [1–4]. The code uses a standard Galerkin formulation with mixed-order interpolation for the velocity and pressure variables. However, a long-standing problem has been the limiting value of Rayleigh number (typically 10^9) beyond which converged results could not be obtained. Difficulties in modelling free convection in a velocity-pressure formulation have been reported by other groups, and the penalty method has been widely adopted as a possible remedy. However, the nature of the penalty method also imposes a limiting value on the Rayleigh number (in finite-precision arithmetic), which is somewhat smaller than that encountered in ENTWIFE.

Although the ENTWIFE limit of around 10^9 is higher than that achieved in most other published finite-element predictions, it falls short of the Rayleigh numbers in the range 10^{11} to 10^{13} that are typical of experiments relevant to fire studies and nuclear reactor thermalhydraulics. As a result, comparison and validation against such experiments has only been possible by extrapolation of the predictions [3–4].

The problem which occurs is characteristic of convection at high Rayleigh number; the flow is concentrated in narrow boundary layers, outside which the fluid stagnates with a high degree of temperature stratification. The equations describing the convection in the stagnant region then reduce to a simple balance between the vertical pressure gradient and the buoyancy force terms, which are of the order of the Rayleigh number in magnitude. Most commonly used elements have too few degrees of freedom to achieve this balance accurately, resulting in mesh-scale oscillations in the vertical velocity. These oscillations increase with Rayleigh number and ultimately prevent convergence unless an unrealistically fine mesh is used in the stagnant region.

To demonstrate this inability to model correctly the balance of buoyancy force and vertical pressure gradient, the equations for free convection in a closed cavity were solved in the Boussinesq approximation, with an imposed temperature distribution in the interior which gives an analytic solution of no flow. Gresho *et al.* [5] have used a similar technique in examining the relative merits of elements. We consider three different cases:

(i) surfaces horizontal–vertical, temperature $T = z$ in cavity, elements aligned with stratification;

(ii) surfaces horizontal–vertical, temperature $T = z^2$ in cavity, elements aligned with stratification;

0021-9991/85 $3.00

581/60/1-11

(iii) surfaces tilted, temperature $T = z$ in cavity, elements not aligned with stratification.

All three cases led to mesh-scale oscillations which increased with Rayleigh number, even though the positive temperature gradient should lead to no flow. This was verified for the following elements and interpolations:

(i) The six-node triangle with quadratic velocities and temperature and linear pressure,

(ii) The seven-node triangle with super-quadratic velocities and temperature and piecewise-linear pressure [6].

(iii) The eight-node quadrilateral with biquadratic velocities and temperature and C^0 bilinear pressure.

(iv) The nine-node quadrilateral with biquadratic velocities and temperature and C^0 bilinear pressure.

The problem was overcome by implementing a different element, a nine-noded quadrilateral with piecewise-linear (discontinuous across element boundaries) variation of the pressure, which has been discussed in [7–10]. The extra pressure freedoms ensure an exact balance between pressure gradient and buoyancy terms, provided the elements are rectangular and parallel to the isotherms. For the first two test cases described, the element did indeed predict successfully the no-flow solution. For the third test case, mesh-scale oscillations resulted, since the elements were not parallel to the isotherms. A fuller discussion of this behaviour is given in the appendix.

With this new element there was no difficulty in simulating convection in rectangular enclosures at Rayleigh numbers up to 10^{12}, using relatively coarse grids. As an example of this, we consider a closed rectangular cavity with a width-to-height ratio of 2.1. The thermal boundary conditions on the closed cavity are

(i) adiabatic left vertical wall;

(ii) right vertical wall at a uniform hot temperature;

(iii) roof at the same uniform hot temperature as the right wall;

(iv) floor at a uniform cold temperature.

Figure 1 shows an irregular grid of 18×18 elements (that is, 37×37 nodes), which is strongly graded to resolve the very narrow boundary layers. Figure 2 shows the predicted streamlines and isotherms obtained on this grid for a Rayleigh number of 10^{12}, based on cavity width, and a Prandtl number of 5.

The use of this element represents a significant breakthrough in the simulation of natural convection in the finite-element method, and opens the way for many new applications, hitherto impossible. The success of the new element is not only confined to convection at high Rayleigh numbers. We have also used it to predict

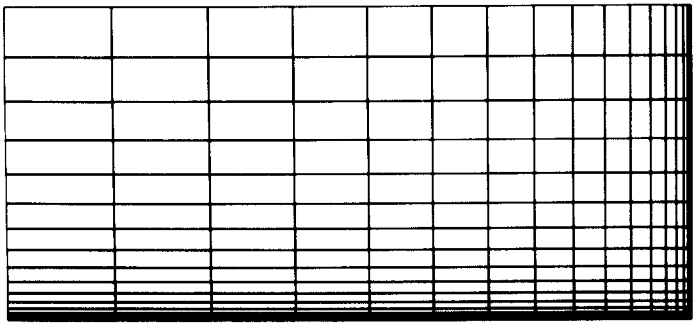

FIGURE 1

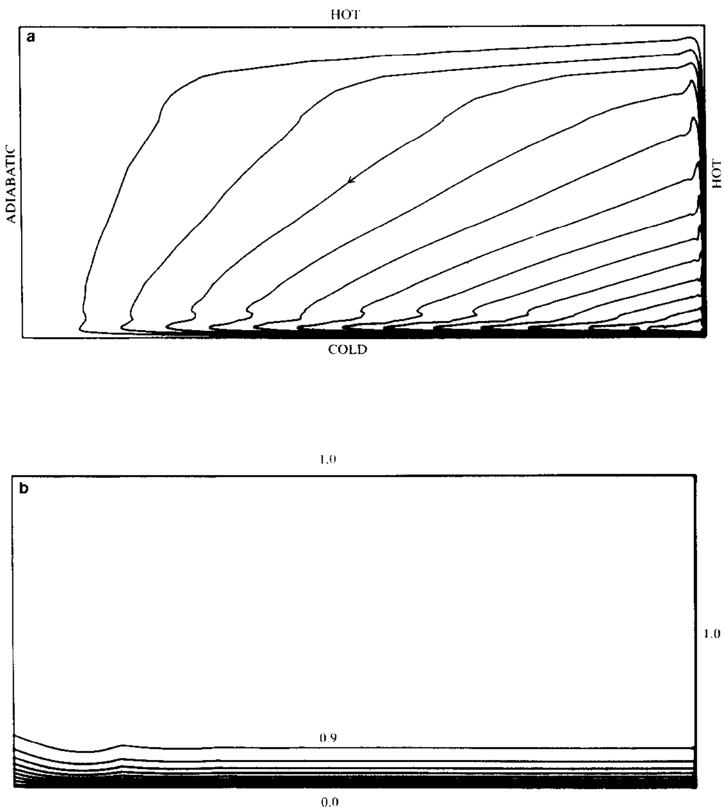

FIGURE 2

values of the critical Rayleigh number for the onset of Bénard convection. These are in accord with the results of other work using an entirely different method. In contrast, the old element was found to give an error of 2% for a grid of identical size.

APPENDIX

We consider the problem of the natural convection of a Boussinesq fluid, with the temperature T a function of the vertical coordinate z only. The analytic solution for the pressure p is

$$p = Ra\, Pr \int T(z)\, dz, \tag{1}$$

where Ra is the Rayleigh number and Pr the Prandtl number. This equation represents simply the balance of the pressure gradient and buoyancy terms. Freedom-counting shows that it is not in general possible to satisfy this requirement if the temperature is modelled by a quadratic interpolation and the pressure by a piecewise-constant, piecewise-linear, linear or quadratic interpolation. However, it may become possible when the temperature field is aligned with elements, on which the basis functions are products of functions of x and z.

We first show that this balance can be achieved using the nine-node quadrilateral with C^{-1} linear interpolation for the pressure. A similar proof holds for the same element with C^{-1} bilinear pressure interpolation, and for the four-node quadrilateral with piecewise constant pressure interpolation.

We have to construct a discrete pressure field such that

$$\int_\Omega p \frac{\partial \psi_i}{\partial x} = 0 \tag{2}$$

and

$$\int_\Omega p \frac{\partial \psi_i}{\partial z} + Ra\, Pr \int_\Omega T\psi_i = 0 \tag{3}$$

for all velocity basis functions ψ_i. Here Ω is the cavity domain.

Let $p = p(z)$ depend only on z and suppose the normal component of velocity is zero on the boundary of the region. Clearly Eq. (2) is satisfied since

$$\int_\Omega p \frac{\partial \psi_i}{\partial x} = \int_0^b \int_0^a p \frac{\partial \psi_i}{\partial x}\, dx\, dz = \int_0^b [p\psi_i]_0^a\, dz = 0.$$

For the element we are considering, we have $\psi_i = A(x)\, B(z)$, thus Eq. (3) becomes

$$\int_0^b \int_0^a p(z)\, A(x) \frac{dB(z)}{dz}\, dx\, dz + Ra\, Pr \int_0^b \int_0^a T(z)\, A(x)\, B(z)\, dx\, dz = 0;$$

that is,

$$\int_0^b p(z)\frac{dB(z)}{dz}\,dz + Ra\,Pr\int_0^b T(z)\,B(z)\,dz = 0. \tag{4}$$

Thus the problem is essentially one dimensional. Number the elements in the vertical direction $j = 1, 2, \ldots, N$, where N is the number of elements. The pressure degrees of freedom in element i are the value p_i and z derivative $p_{z,i}$ at the centre-node of the element. Equation (4) then reduces to $2N - 1$ equations involving the pressure freedoms as follows:

$$\tfrac{2}{3}\,\Delta z_i p_{z,i} = t_i, \tag{5}$$

$$2(p_i - p_{i+1}) + \tfrac{2}{3}\,\Delta z_i(p_{z,i} + p_{z,i+1}) = t_{i+1/2}, \tag{6}$$

where Δz_i is the height of the i^{th} element and t_i and $t_{i+1/2}$ depend only on $T(z)$. It is now clear that Eqs. (5) and (6) always have a solution, for any $T(z)$, which is unique up to an additional constant (as usual for the pressure in incompressible flow).

The above proof is almost trivial but it is important to note that it depends on two points. First the element must have enough pressure degrees of freedom, and second a rectangular mesh aligned with the isotherms in the stratified region must be used.

Thus the seven-noded triangle with augmented quadratic interpolation for velocity and piecewise-linear interpolation for pressure performs badly, because its basis functions do not factorize, even though it has many more pressure degrees of freedom than the six-noded triangle. In fact this particular element has one degree of freedom per vector momentum equation, and so has the optimum constraint ratio as defined by Gresho *et al.* [5]. Nevertheless, it is incapable of modelling convection at very high Rayleigh numbers in the velocity-pressure formulation.

References

1. K. H. Winters, "A Numerical Study of Natural Convection in a Square Cavity," *Harwell Report* AERE R.9747, 1980.
2. C. P. Jackson and K. H. Winters, "A Finite-Element Study of the Bénard Problem Using Parameter-Stepping and Bifurcation Search," *Harwell Report* AERE TP.936, 1982; *Int. J. Numer. Methods Fluids* 4 (1984), 127–145.
3. K. H. Winters, "Predictions of Laminar Natural Convection in Heated Cavities," *Harwell Report* AERE TP.942, 1982; "*Numerical Methods in Heat Transfer*" R. W. Lewis, K. Morgan and B. A. Schrefler, Eds.), Vol. 2, pp. 179–204, Wiley, Chichester, 1983.
4. P. M. Birchenough and K. H. Winters, "Finite-Element Predictions of Laminar Natural Convection in a Trapezoidal Water-Filled Plenum and Comparison with Experiment," *Harwell Report* AERE TP.952, 1982; "Proceedings, *Second International Conference on Nuclear Reactor Thermalhydraulics*, Santa Barbara," pp. 1403–1411, 1983.
5. P. M. Gresho, R. L. Lee, S. T. Chan, and J. M. Leone, Jr., *in "Third International Conference on Finite Element Methods in Flow Problems*, Banff," 1980.

6. M. Crouzeix and P. A. Raviart, *RAIRO, Ser. Anal. Numero* **3** (1973), 33.
7. D. F. Griffiths, *Int. J. Numer. Methods Fluids* **1** (1981), 323.
8. M. Fortin, *Int. J. Numer. Methods Fluids* **1** (1981), 347.
9. M. Engelman, R. Sani, P. Gresho, and M. Bercovier, *Int. J. Numer. Methods Fluids* **2** (1982), 25.
10. M. J. P. Cullen, *J. Comput. Phys.* **51** (1983), 273.

Received: March 21, 1984; revised: August 14, 1984

K. A. Cliffe
C. P. Jackson
K. H. Winters

Theoretical Physics Division
AERE Harwell, Didcot
Oxon OX11 ORA, England